

## A PRIORI $L^\rho$ ERROR ESTIMATES FOR GALERKIN APPROXIMATIONS TO POROUS MEDIUM AND FAST DIFFUSION EQUATIONS

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ABSTRACT. Galerkin approximations to solutions of a Cauchy-Dirichlet problem governed by the generalized porous medium equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} (|u|^{\rho-2} \frac{\partial u}{\partial x_i}) = f(x, t)$$

on bounded convex domains are considered. The range of the parameter  $\rho$  includes the fast diffusion case  $1 < \rho < 2$ . Using an Euler finite difference approximation in time, the semi-discrete solution is shown to converge to the exact solution in  $L^\infty(0, T; L^\rho(\Omega))$  norm with an error controlled by  $O(\Delta t^{\frac{1}{4}})$  for  $1 < \rho < 2$  and  $O(\Delta t^{\frac{1}{2\rho}})$  for  $2 \leq \rho < \infty$ . For the fully discrete problem, a global convergence rate of  $O(\Delta t^{\frac{1}{4}})$  in  $L^2(0, T; L^\rho(\Omega))$  norm is shown for the range  $\frac{2N}{N+1} < \rho < 2$ . For  $2 \leq \rho < \infty$ , a rate of  $O(\Delta t^{\frac{1}{2\rho}})$  is shown in  $L^\rho(0, T; L^\rho(\Omega))$  norm.

### 1. INTRODUCTION

Consider the Cauchy-Dirichlet problem governed by the generalized porous medium equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( |u|^{\rho-2} \frac{\partial u}{\partial x_i} \right) &= f(x, t), & (x, t) \in \Omega \times [0, T], \\ (1.1) \quad u(x, t) &= 0, & (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned}$$

where  $1 < \rho < \infty$ ,  $0 < T < \infty$ , and  $\Omega$  is a bounded convex polygonal domain in  $\mathbb{R}^N$ . The above equation is one of the simplest and best-known nonlinear equations of degenerate ( $\rho > 2$ ) or singular ( $\rho < 2$ ) parabolic type. We refer to Kalashnikov [15] and the references therein for background of the basic theory and for applications which motivate the study of this equation. The case  $\rho \geq 2$  is the classical porous medium equation. The case  $1 < \rho < 2$  is referred to as the fast diffusion equation and includes applications in plasma physics [1] and diffusion of impurities in silicon [13]. In this paper we will establish error estimates for both semi-discrete and fully discrete approximations to solutions of (1.1) for a range of  $\rho$  which includes both

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porous medium and fast diffusion behavior. We employ a backward Euler difference approximation in  $t$  and a Galerkin finite element approximation in  $x$ .

In the first of our main results we obtain error estimates for semi-discrete approximations over the full range  $1 < \rho < \infty$  in the  $L^\infty(0, T; L^\rho(\Omega))$  norm. In particular, we establish a convergence rate of  $O(\Delta t^{\frac{1}{4}})$  for  $1 < \rho < 2$  and  $O(\Delta t^{\frac{1}{2\rho}})$  for  $2 \leq \rho < \infty$ . This result can be compared to work of Rulla [25]. His analysis covers the semi-discrete problem for the full range of  $\rho$ . In the fast diffusion case, Rulla obtains a better rate of  $O(\Delta t)$  in the space  $L^\infty(0, T; H^{-1}(\Omega))$ . He also obtains the rate  $O(\Delta t)$  for the porous medium case using the norm  $L^2(0, T; L^2(\Omega))$ . Rulla's results apply to a general class of maximal monotone operators which can be written as a subgradient. Another result for the semi-discrete approximation to solutions of (1.1) in the fast diffusion case is obtained by Eden, Michaux, and Rakotoson [8, Theorem 6.1]. They establish an  $L^\infty(0, T; H^{-1}(\Omega))$  error estimate of  $O(\Delta t^{\frac{1}{2}})$  under the assumption that  $u_0 \in L^\infty$ . We also mention a related paper by LeRoux [16] where she constructs approximate solutions using a semi-discrete scheme for a fast diffusion problem. Of interest is that her discretized solutions exhibit the same phenomenon known to hold for true solutions in the fast diffusion case, namely that there exists a finite extinction time  $T$  after which the solution is zero. For other related results, see Farago [10] and Garcia [11].

We also establish error estimates for fully discrete approximations in this work by using estimates of the form

$$\max_{1 \leq i \leq m} \frac{\|U_i - U_{i-1}\|_\rho}{\Delta t^{\frac{1}{\rho}}} \leq C \quad \text{for } 2 \leq \rho < \infty,$$

and

$$\max_{1 \leq i \leq m} \frac{\|U_i - U_{i-1}\|_\rho}{\Delta t^{\frac{1}{2}}} \leq C \quad \text{for } 1 < \rho < 2,$$

where  $U_i$  is the fully discrete approximation. In particular, let

$$\rho^* = \frac{2N}{N + 1}.$$

We show that the global convergence rate is  $O(\Delta t^{\frac{1}{4}})$  for  $\rho^* < \rho < 2$  and  $O(\Delta t^{\frac{1}{2\rho}})$  for  $2 \leq \rho < \infty$ . These rates are obtained by taking the spatial mesh size  $h = O(\Delta t^\beta)$ , where  $\beta = \frac{\rho}{\rho(N+2)-2N}$  and  $\beta = \frac{(\rho-1)\rho}{\rho(N+2)-2N}$ , respectively. Note that these error estimates are in the space  $L^2(0, T; L^\rho(\Omega))$  when  $\rho^* < \rho < 2$  and  $L^\rho(0, T; L^\rho(\Omega))$  when  $2 \leq \rho < \infty$ .

Our fully discrete result compares to a recent paper by Rulla and Walkington [26] where the optimal rate of  $O(\Delta t)$  is proved for two-dimensional problems in the norm  $L^\infty(0, T; H^{-1}(\Omega))$ . Rulla and Walkington also obtain  $L^2(0, T; L^2(\Omega))$  estimates for the classical porous medium case  $2 \leq \rho < \infty$ . We note that in order to obtain the appropriate  $L^\infty$  bounds in [26] their analysis is restricted to two dimensions. A fully discrete error analysis for a closely related fast diffusion problem is studied by Lesaint and Pousin in [17]. They consider the problem  $vv_t - v_{xx} = 0$ , which is a one-dimensional version of (1.1) with  $\rho = 3/2$  after the change of variable  $u = v^2$ . Assuming nonzero boundary data, they obtain a  $O(\frac{\Delta t}{h} + h^{1/2})$  error estimate in  $L^\infty(0, T; L^2(\Omega))$  norm.

An error analysis for the case  $\rho > 2$  was done by Rose [24] who worked on the Neumann problem for the porous medium equation. A related effort by Jerome

and Rose [14] considered the Stefan problem with Neumann boundary conditions. These results were greatly extended and improved by Nocketto, Verdi, and Elliott [9], [19], [20], [29], who developed a theory which covers a wide class of singular and degenerate problems under rather general assumptions about the initial and boundary data. Further references can be found in these papers.

A key idea used in [24] to study the fully discretized porous medium equation is to regularize the original equation by considering the nondegenerate problem

$$(1.2) \quad \frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( g_\varepsilon(u) \frac{\partial u}{\partial x_i} \right) = f(x, t),$$

where  $g_\varepsilon(x) = |x|^{\rho-2}x$  outside a small neighborhood of  $x = 0$  and  $g_\varepsilon(x) \geq \varepsilon > 0$  near the origin. The solution  $u_\varepsilon$  of (1.2) is then approximated using finite differences in time and finite elements in space. Error estimates in terms of the regularization parameter  $\varepsilon$ , the time step  $\Delta t$ , and the finite element size  $h$  for the fully discrete problem can then be obtained. By choosing an appropriate relationship between these parameters, a global rate of convergence can be established in terms of  $\Delta t$  or  $h$ . The work of Nocketto and Verdi [19], [20] also uses a regularizing perturbation to establish a global rate of convergence in the  $L^\infty(0, T; H^{-1}(\Omega))$  norm of  $O(\Delta t^{\frac{1}{2}})$  by taking  $h = O(\Delta t^{\frac{\rho-1}{\rho}})$ ,  $\varepsilon = O(\Delta t^{\frac{\rho-2}{\rho}})$  and assuming solutions are in  $L^\infty(0, T; L^\infty(\Omega))$ . In addition to the  $L^\infty(0, T; H^{-1}(\Omega))$  error estimates, they derive a rate of  $O(\Delta t^{\frac{1}{\rho}})$  in the  $L^\infty(0, T; L^\rho(\Omega))$  norm under the assumption that

$$\text{meas}\{x \in \Omega : 0 < u(x) < \varepsilon^{\frac{1}{\rho-2}}\} \leq C\varepsilon,$$

which is shown to be true for  $\dim(\Omega) = 1$  in [3]. It is important to point out that the analysis in [20] includes the errors induced by numerical integration.

The regularizing approximation (1.2) is avoided in [29] where  $L^\infty(0, T; H^{-1}(\Omega))$  error estimates are derived for the fully discrete approximation of a general class of monotone operators. Using numerical integration,  $C^0$  piecewise linear finite elements in space, and backward differences in time, a global rate of  $O(\Delta t^{\frac{1}{2}})$  in the  $L^\infty(0, T; H^{-1}(\Omega))$  norm is established with  $h = O(\Delta t)$ . We note that this result assumes  $f = 0$  and initial data  $u_0 \in L^2$ . Verdi's analysis is clearly applicable to the range  $2 \leq \rho < \infty$ . Moreover, from [21] it appears that by combining the results of [9], [20], [29], Verdi's work can be modified to cover the fast diffusion case for  $\rho^* < \rho < 2$  in the  $L^\infty(0, T; H^{-1}(\Omega))$  norm with the same rate  $O(\Delta t^{\frac{1}{2}})$ .

This paper unifies results for finite element error estimates for solutions of (1.1) covering both the fast diffusion case and the porous medium case. We avoid any regularizing approximation; however, our work does not include numerical integration as in [20] and [29]. We prove our results under Dirichlet boundary conditions, and assume that the initial function  $u_0$  satisfies  $|u_0|^{\rho-2}u_0 \in H_0^1 \cap L^{\rho'}$ . In particular, we do not assume the solution  $u$  is in  $L^\infty$ . Our global convergence rates are not as strong as the optimal ones obtained in [25] and [26], but we are working in the space  $L^\rho$  which has a more practical norm than  $H^{-1}$ . We also provide explicit proofs which extend fast diffusion results that exist only implicitly in the literature.

The paper is organized as follows. In Section 2 we state existence, uniqueness, and other preliminary results. In Section 3 we derive regularity estimates and error estimates for the semi-discrete approximations. In Section 4 we treat the fully discrete problem.

2. PRELIMINARIES

Throughout this work, we assume that  $f : [0, T] \rightarrow L^2$  is Lipschitz continuous, i.e., there exists  $L > 0$  such that  $\|f(t) - f(t')\|_2 \leq L|t - t'|$ . Existence and uniqueness of solutions to (1.1) has been studied by many authors (see, e.g., Raviart [22], [23], Lions [18], Tsutsumi [28], and the references therein). In particular, it can be shown using the abstract theory of evolution equations governed by accretive operators (see, e.g., Brézis and Crandall [2], and Brézis and Friedman [3]) that if  $u_0 \in L^1$ , then there is a unique solution to (1.1) in  $C([0, T], L^1)$ . We say that a function  $u$  satisfies (1.1) if

$$(2.1) \quad \int_0^T \left\langle \frac{du}{dt}, v \right\rangle + \frac{1}{\rho - 1} \langle \nabla(|u|^{\rho-2}u), \nabla v \rangle dt = \int_0^T \langle f, v \rangle dt,$$

for all  $v \in L^\rho(0, T; H_0^1 \cap L^{\rho'})$ , where  $\langle f, g \rangle = \int f(x)g(x) dx$ .

By (2.1) and a proof similar to that in Ženíšek [30, Theorem 43.3], we have

$$(2.2) \quad \left\langle \frac{\partial u}{\partial t}, v \right\rangle + \frac{1}{\rho - 1} \langle \nabla(|u|^{\rho-2}u), \nabla v \rangle = \langle f, v \rangle, \quad \forall v \in H_0^1 \cap L^{\rho'},$$

for almost every  $t$  in  $[0, T]$ . Henceforth, we shall use (2.2) without repeating that it holds for almost every  $t$  in  $[0, T]$ .

Let  $\Phi(x) = |x|^{\rho-2}x$ , where  $x \in \mathbb{R}^1$ . Note that  $\Phi^{-1}(x) = |x|^{\rho'-2}x$ , where  $\rho' = \frac{\rho}{\rho-1}$ . The following estimates hold for all  $x, y \in \mathbb{R}^1$ ; the constant  $C > 0$  is independent of  $x$  and  $y$  (for proofs, see DiBenedetto [6]).

$$(2.3) \quad \begin{aligned} \text{For } 2 \leq \rho < \infty, \quad &|x - y|^\rho \leq C(\Phi(x) - \Phi(y))(x - y), \\ &|\Phi(x) - \Phi(y)| \leq C|x - y|(|x| + |y|)^{\rho-2}. \end{aligned}$$

$$(2.4) \quad \begin{aligned} \text{For } 1 < \rho < 2, \quad &|x - y|^2 \leq C(\Phi(x) - \Phi(y))(x - y)(|x| + |y|)^{2-\rho}, \\ &|\Phi(x) - \Phi(y)| \leq C|x - y|^{\rho-1}. \end{aligned}$$

Throughout this paper we assume our initial data satisfies  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . In particular, since  $|\Phi(u_0)| = |u_0|^{\rho-1}$ , it is clear that  $\Phi(u_0) \in L^{\rho'}$  if and only if  $u_0 \in L^\rho$ . Thus, if  $\rho \leq 2$  we have  $u_0 \in L^2$ . Also, since  $p^* < \rho < 2$  implies  $1 < \frac{2}{\rho-1} < \frac{2N}{N-2}$ , we can conclude from the Sobolev imbedding that  $\Phi(u_0) \in L^{\frac{2}{\rho-1}}$  and hence  $u_0 \in L^2$  in this case also. Thus, our assumption on initial data implies  $u_0 \in L^\rho \cap L^2$  provided  $p^* < \rho < \infty$ .

We use capital  $C$ 's for generic positive constants and the standard  $L^\rho$  norms are denoted by  $\|\cdot\|_\rho$ . We use  $\|\cdot\|_{-1}$  to denote the norm in  $H^{-1}$ , the dual space of  $H_0^1$ . All integrals and function spaces will be over the domain  $\Omega$  unless otherwise noted. We use a dot to denote the time derivative, e.g.,  $\dot{u} = \frac{du}{dt}$ . For  $\psi \in H_0^1$  and  $g \in H^{-1}$  we can write the duality pairing  $(g, \psi)_{H^{-1}} = \langle \nabla(-\Delta)^{-1}g, \nabla\psi \rangle$ . In light of this, we henceforth simplify our notation by writing  $(g, \psi)_{H^{-1}}$  as  $\langle g, \psi \rangle$ . The appropriate inner product will be clear from context.

3. ERROR ESTIMATES FOR THE SEMI-DISCRETE APPROXIMATION

Let  $\{t_i\}_{i=0}^m$  be a partition of the interval  $[0, T]$ , i.e.,  $0 = t_0 < t_1 < t_2 < \dots < t_m = T$  and  $\Delta t_i = t_i - t_{i-1}$  for  $i = 1, \dots, m$ . We denote the mesh of this partition by  $\Delta t = \max_{1 \leq i \leq m} \Delta t_i$ . We construct a sequence  $\{u_i\}_{i=0}^m$  by solving the following recurrence nonlinear elliptic problem. Given  $u_{i-1}$ , find  $u_i$  such that

$$(3.1) \quad \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, v \right\rangle + \frac{1}{\rho - 1} \langle \nabla(|u_i|^{\rho-2} u_i), \nabla v \rangle = \langle f_i, v \rangle, \quad \forall v \in H_0^1 \cap L^{\rho'}$$

where  $u_0 = u_0(x)$ ,  $f_i = f(x, t_i)$ ,  $i = 1, \dots, m$ .

We first show that solutions  $u_i$  exist for (3.1). Consider the following auxiliary recurrence problem obtained from (3.1) by writing  $v_i = \Phi(u_i)$ . Given  $v_{i-1}$ , find  $v_i$  such that

$$(3.2) \quad \left\langle \frac{|v_i|^{\rho'-2} v_i - |v_{i-1}|^{\rho'-2} v_{i-1}}{\Delta t_i}, v \right\rangle + \frac{1}{\rho - 1} \langle \nabla v_i, \nabla v \rangle = \langle f_i, v \rangle,$$

for  $v \in H_0^1 \cap L^{\rho'}$ , where  $i = 1, \dots, m$ , and  $v_0 = \Phi(u_0)$ . The operator defined by

$$v \rightarrow |v|^{\rho'-2} v - K \Delta v, \quad \text{where } K > 0,$$

is bounded, hemicontinuous, strictly monotone and coercive from  $H_0^1 \cap L^{\rho'}$  to  $H^{-1} + L^{\rho}$ , which is the dual space of  $H_0^1 \cap L^{\rho'}$  [22]. Therefore, a unique sequence  $\{v_i\}_{i=1}^m$  in  $H_0^1 \cap L^{\rho'}$  can be generated from (3.2) and the standard theory of monotone operators (see Browder [4]) with the assumption that  $v_0 = \Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . Now since  $u_i = \Phi^{-1}(v_i)$  for  $i = 1, \dots, m$ , we conclude that the sequence  $\{u_i\}_{i=1}^m$  satisfies (3.1). Note that  $v \in L^{\rho'}$  if and only if  $u = \Phi^{-1}(v) \in L^{\rho}$ . The semi-discrete solution of (1.1) is defined, for a given partition  $\{t_i\}_{i=1}^m$ , by linear interpolation in  $L^{\rho}$ , that is

$$(3.3) \quad u_m(t) = \frac{t - t_{i-1}}{\Delta t_i} u_i + \frac{t_i - t}{\Delta t_i} u_{i-1}, \quad \text{for } t_{i-1} < t \leq t_i, \quad i = 1, \dots, m,$$

$$(3.4) \quad u_m(0) = u_0.$$

We observe  $u_m(t) - u_i = \frac{(t-t_i)}{\Delta t_i} (u_i - u_{i-1})$  and

$$\frac{du_m(t)}{dt} = \frac{u_i - u_{i-1}}{\Delta t_i}, \quad \text{for } t_{i-1} < t \leq t_i, \quad i = 1, \dots, m.$$

**Definition 3.1.** We say that a partition  $0 = t_0 < t_1 < t_2 < \dots < t_m = T$  with  $\Delta t_i = t_i - t_{i-1}$  is nonincreasing if it satisfies  $\Delta t_i \leq \Delta t_{i-1}$  for  $i = 2, \dots, m$ .

**Lemma 3.1.** Suppose that  $1 < \rho < \infty$  and the partition  $\{t_i\}_{i=0}^m$  is a nonincreasing partition of  $[0, T]$  as defined above. Let  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . Suppose  $\{u_i\}_{i=1}^m$  is the sequence generated by (3.1) and let  $u_m(t)$  be the corresponding semi-discrete solution defined by (3.3), (3.4). Then there exists a positive constant  $C = C(\Omega, \rho, f, u_0)$ , independent of  $\{t_i\}_{i=0}^m$ , such that

$$\begin{aligned} \max_{1 \leq i \leq m} \|u_i\|_{\rho} &\leq C, \\ \max_{1 \leq i \leq m} \|\nabla \Phi(u_i)\|_2 &\leq C, \end{aligned}$$

and

$$\max_{0 \leq t \leq T} \left\| \frac{du_m(t)}{dt} \right\|_{-1} \leq C.$$

*Proof.* Fix  $1 \leq i \leq m$  and let  $v = v_i$  in (3.2). We have

$$(3.5) \quad \left\langle \frac{|v_i|^{\rho'-2}v_i - |v_{i-1}|^{\rho'-2}v_{i-1}}{\Delta t_i}, v_i \right\rangle + \frac{1}{\rho - 1} \langle \nabla v_i, \nabla v_i \rangle = \langle f_i, v_i \rangle.$$

Using (3.5) and the following inequality (see [22]),

$$\frac{1}{\rho} (\|v_i\|_{\rho'}^{\rho'} - \|v_{i-1}\|_{\rho'}^{\rho'}) \leq \langle |v_i|^{\rho'-2}v_i - |v_{i-1}|^{\rho'-2}v_{i-1}, v_i \rangle,$$

we get

$$\frac{1}{\Delta t_i} (\|v_i\|_{\rho'}^{\rho'} - \|v_{i-1}\|_{\rho'}^{\rho'}) + \rho' \|\nabla v_i\|_2^2 \leq \rho \langle f_i, v_i \rangle.$$

We have, for any  $\varepsilon > 0$ ,  $|\langle f_i, v_i \rangle| \leq \frac{1}{2\varepsilon} \|f_i\|_{-1}^2 + \frac{\varepsilon}{2} \|\nabla v_i\|_2^2$ ; therefore,

$$(3.6) \quad \|v_i\|_{\rho'}^{\rho'} - \|v_{i-1}\|_{\rho'}^{\rho'} + \left(\rho' - \frac{\varepsilon\rho}{2}\right) \Delta t_i \|\nabla v_i\|_2^2 \leq \frac{\rho\Delta t_i}{2\varepsilon} \|f_i\|_{-1}^2.$$

In (3.6), using the Sobolev inequality  $\|f_i\|_{-1} \leq C\|f_i\|_2$  and summing, we get

$$(3.7) \quad \|v_i\|_{\rho'}^{\rho'} + \left(\rho' - \frac{\varepsilon\rho}{2}\right) \sum_{s=1}^i \Delta t_s \|\nabla v_s\|_2^2 \leq \frac{\rho C}{2\varepsilon} \sum_{s=1}^i \Delta t_s \|f_s\|_2^2 + \|v_0\|_{\rho'}^{\rho'}.$$

By choosing  $0 < \varepsilon < \frac{2}{\rho-1}$  in (3.7), we find that both terms on the left are nonnegative; thus,  $\|v_i\|_{\rho'} \leq C(f, v_0)$  and  $\sum_{s=1}^i \Delta t_s \|\nabla v_s\|_2^2 \leq C(f, v_0)$ . The first inequality in Lemma 3.1 follows since  $\|v\|_{\rho'}^{\rho'} = \|u\|_{\rho}^{\rho}$ .

Letting  $v = v_i - v_{i-1}$  in (3.2) for  $i = 1, \dots, m$  and using (2.3) and (2.4), we have

$$0 \leq (|v_i|^{\rho'-2}v_i - |v_{i-1}|^{\rho'-2}v_{i-1})(v_i - v_{i-1}).$$

Therefore,

$$\frac{1}{\rho - 1} \langle \nabla v_i, \nabla v_i - \nabla v_{i-1} \rangle \leq \langle f_i, v_i - v_{i-1} \rangle,$$

which gives

$$\|\nabla v_i\|_2^2 - \|\nabla v_{i-1}\|_2^2 - 2(\rho - 1)(\langle f_i, v_i \rangle - \langle f_{i-1}, v_{i-1} \rangle) \leq 2(\rho - 1)\langle f_{i-1} - f_i, v_{i-1} \rangle.$$

Summing this inequality from 1 to  $i$  and using the Lipschitz continuity of  $f$  we get

$$\|\nabla v_i\|_2^2 - \|\nabla v_0\|_2^2 - 2(\rho - 1)(\langle f_i, v_i \rangle - \langle f_0, v_0 \rangle) \leq 2(\rho - 1)L \sum_{s=1}^i \Delta t_s \|v_{s-1}\|_2,$$

which implies

$$(3.8) \quad \|\nabla v_i\|_2^2 \leq C_1 \|\nabla v_i\|_2 + C_2 \sum_{s=1}^i \Delta t_s \|v_{s-1}\|_2 + C_3,$$

where  $C_1, C_2$ , and  $C_3$  depend only on  $\rho, f$  and  $v_0$ . Since  $\Delta t_s \leq \Delta t_{s-1}$  for  $s \geq 2$  and  $\sum_{s=1}^i \Delta t_s \|\nabla v_s\|_2^2 \leq C(f, v_0)$  we have

$$\begin{aligned} \sum_{s=1}^i \Delta t_s \|v_{s-1}\|_2 &\leq \Delta t_1 \|v_0\|_2 + \sum_{s=2}^i \Delta t_{s-1} \|v_{s-1}\|_2 \\ &\leq \Delta t_1 \|v_0\|_2 + \left( \sum_{s=2}^i \Delta t_{s-1} \|\nabla v_{s-1}\|_2^2 \right)^{\frac{1}{2}} \left( \sum_{s=2}^i \Delta t_{s-1} \right)^{\frac{1}{2}} \\ &\leq C(f, v_0), \end{aligned}$$

and we deduce from (3.8) that

$$(3.9) \quad \|\nabla v_i\|_2^2 \leq C_1 \|\nabla v_i\|_2 + C_2.$$

From (3.9) we conclude that there exists a positive constant  $C(f, v_0)$  such that  $\|\nabla v_i\|_2 \leq C(f, v_0)$ . Thus  $\|\nabla \Phi(u_i)\|_2 \leq C(f, v_0)$  which gives the second inequality in Lemma 3.1.

To prove the third inequality, we use (3.1) and write

$$\left| \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, v \right\rangle \right| \leq \left| -\frac{1}{\rho - 1} \langle \nabla(|u_i|^{\rho-2} u_i), \nabla v \rangle + (f_i, v) \right|, \quad \forall v \in H_0^1 \cap L^{\rho'}.$$

Again using that  $\|f_i\|_{-1} \leq C\|f_i\|_2$  we estimate

$$(3.10) \quad \left| \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, v \right\rangle \right| \leq \left( \frac{1}{\rho - 1} \|\nabla v_i\|_2 + C\|f_i\|_2 \right) \|\nabla v\|_2.$$

The third inequality now follows from (3.10) since  $\|\nabla v_i\|_2$  and  $\|f_i\|_2$  are both bounded. □

We suspect our hypothesis that the partition of  $[0, T]$  be nonincreasing is only an artifact of our method of proof and could perhaps be relaxed to include arbitrary partitions. However, we have not found a way to estimate (3.8) without this assumption. Note that many related results (e.g., Verdi [24]) assume a uniform mesh which is clearly nondecreasing.

Using the above  $H^{-1}$  estimate on  $\frac{du_m(t)}{dt}$  we can write  $\|u_i - u_{i-1}\|_{-1} \leq C\Delta t_i$ . Let  $1 \leq l \leq m$  and sum from  $i = 1, \dots, l$  to conclude  $\|u_l\|_{-1} - \|u_0\|_{-1} \leq CT$ . We conclude

$$(3.11) \quad \|u_m(t)\|_{-1} \leq C,$$

where  $C$  is independent of  $\Delta t$ .

**Theorem 3.1.** *Let  $u$  be the exact solution of (1.1) with initial data satisfying  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . And let  $u_m(t)$  be the semi-discrete solution defined by (3.3), (3.4). Then*

$$\|u(t) - u_m(t)\|_{-1} \leq C\Delta t^{\frac{1}{2}}, \quad \text{for } 1 < \rho < \infty.$$

*Proof.* Let  $\{t_i\}_{i=0}^m$  and  $\{t_{i'}\}_{i'=0}^{m'}$  be two partitions of the interval  $[0, T]$ . Let  $u_m(t)$  and  $u_{m'}(t)$  be the semi-discrete solutions defined by (3.3) and (3.4) corresponding to the partitions, respectively. For each  $g \in H^s$ ,  $s \geq -1$ , let  $Tg$  denote the unique solution  $u$  of  $-\Delta u = g$  in  $H_0^1$ . Then  $\|Tg\|_2 \leq C\|g\|_2$  (see, e.g., Gilbarg and Trudinger [12]).

Let  $e_{m,m'}(t) = u_m(t) - u_{m'}(t)$ , and define  $\rho_{m,m'}(t) \in H_0^1$  by  $-\Delta\rho_{m,m'}(t) = e_{m,m'}(t)$ . Then by (3.1) we have, for  $t \in (t_{i-1}, t_i] \cap (t_{i'-1}, t_{i'}]$ ,

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\nabla\rho_{m,m'}(t)\|_2^2 &= \left\langle \frac{d}{dt}(\nabla\rho_{m,m'}(t)), \nabla\rho_{m,m'}(t) \right\rangle \\
 &= \left\langle \frac{d}{dt}(-\Delta\rho_{m,m'}(t)), \rho_{m,m'}(t) \right\rangle \\
 (3.12) \quad &= \left\langle \frac{du_m(t)}{dt} - \frac{du_{m'}(t)}{dt}, Te_{m,m'}(t) \right\rangle \\
 &= \langle f_i - f_{i'}, Te_{m,m'}(t) \rangle \\
 &\quad - \frac{1}{\rho - 1} \langle \Phi(u_i) - \Phi(u_{i'}), e_{m,m'}(t) \rangle.
 \end{aligned}$$

Using (3.11), the Lipschitz continuity of  $f$ , and the fact that  $-\Delta w = e_{m,m'}$  implies  $\|\nabla w\|_2^2 \leq \|e_{m,m'}\|_{-1} \|\nabla w\|_2$ , we have

$$\begin{aligned}
 |\langle f_i - f_{i'}, Te_{m,m'}(t) \rangle| &\leq \|f_i - f_{i'}\|_{-1} \|\nabla Te_{m,m'}(t)\|_2 \\
 (3.13) \quad &\leq C \|f_i - f_{i'}\|_2 \|e_{m,m'}(t)\|_{-1} \\
 &\leq LC |t_i - t_{i'}|.
 \end{aligned}$$

Now, since  $\langle \Phi(u_i) - \Phi(u_{i'}), u_i - u_{i'} \rangle \geq 0$ , we derive from (3.12) and (3.13) that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\nabla\rho_{m,m'}(t)\|_2^2 &\leq LC |t_i - t_{i'}| \\
 (3.14) \quad &\quad + \frac{1}{\rho - 1} |\langle \Phi(u_i) - \Phi(u_{i'}), u_m(t) - u_i + u_{i'} - u_{m'}(t) \rangle|.
 \end{aligned}$$

We also have

$$\begin{aligned}
 &|\langle \Phi(u_i) - \Phi(u_{i'}), u_m(t) - u_i + u_{i'} - u_{m'}(t) \rangle| \\
 &= \left| \left\langle \Phi(u_i) - \Phi(u_{i'}), \frac{t - t_i}{\Delta t_i} (u_i - u_{i-1}) - \frac{t - t_{i'}}{\Delta t_{i'}} (u_{i'} - u_{i'-1}) \right\rangle \right| \\
 (3.15) \quad &\leq \|\nabla(\Phi(u_i) - \Phi(u_{i'}))\|_2 \\
 &\quad \times \left( \frac{|t_i - t|}{\Delta t_i} \|u_i - u_{i-1}\|_{-1} + \frac{|t_{i'} - t|}{\Delta t_{i'}} \|u_{i'} - u_{i'-1}\|_{-1} \right).
 \end{aligned}$$

Therefore, by (3.14), (3.15) and Lemma 3.1, we get

$$\frac{1}{2} \frac{d}{dt} \|\nabla\rho_{m,m'}(t)\|_2^2 \leq C_1 |t_i - t| + C_2 |t_{i'} - t| \leq C(\Delta t_i + \Delta t_{i'}),$$

which implies that

$$(3.16) \quad \|\nabla\rho_{m,m'}(t)\|_2^2 \leq C(\Delta t + \Delta t').$$

Since  $-\Delta\rho_{m,m'}(t) = e_{m,m'}(t)$ , we have  $\|e_{m,m'}(t)\|_{-1} \leq \|\nabla\rho_{m,m'}(t)\|_2$ . Therefore, by (3.16), we have

$$(3.17) \quad \|e_{m,m'}(t)\|_{-1}^2 \leq C(\Delta t + \Delta t').$$

From (3.17) we see that for  $1 < \rho < \infty$ ,  $\{u_m(t)\}_{m=1}^\infty$  is Cauchy in  $C(0, T; H^{-1})$  and  $u_m(t) \rightarrow u(t)$  for some  $u(t)$  in  $C(0, T; H^{-1})$  as  $m \rightarrow \infty$ . This  $u(t)$  is the unique solution of the Cauchy-Dirichlet problem. Taking the limit as  $\Delta t' \rightarrow 0$  (and hence  $m' \rightarrow \infty$ ) in (3.17) gives the estimates of Theorem 3.1.  $\square$



**Lemma 3.2.** *Let  $u$  be the exact solution (1.1) with initial data satisfying  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . Let  $\bar{u}_m(t)$  be the piecewise constant approximation (in time) defined by*

$$\bar{u}_m(t) = u_i, \quad \text{for } t_{i-1} \leq t_i, \quad i = 1, \dots, m, \quad \bar{u}_m(0) = u_0.$$

Then

$$\begin{aligned} \|u(t) - \bar{u}_m(t)\|_{\rho} &\leq C\Delta t^{\frac{1}{2\rho}}, \quad \text{for } 2 \leq \rho < \infty, \\ \|u(t) - \bar{u}_m(t)\|_{\rho} &\leq C\Delta t^{\frac{1}{4}}, \quad \text{for } 1 < \rho < 2. \end{aligned}$$

*Proof.* Let  $e_{m,m'}(t) = u_m(t) - u_{m'}(t)$  and  $\bar{e}_{m,m'}(t) = u_i - u_{i'}$  for  $t \in (t_{i-1}, t_i] \cap (t_{i'-1}, t_{i'}]$ . In this interval we have

$$\begin{aligned} &\|e_{m,m'}(t) - \bar{e}_{m,m'}(t)\|_{-1} \\ &\leq \left\| \frac{t_i - t}{\Delta t_i} (u_i - u_{i-1}) \right\|_{-1} + \left\| \frac{t_{i'} - t}{\Delta t_{i'}} (u_{i'} - u_{i'-1}) \right\|_{-1} \\ &= |t_i - t| \left\| \frac{du_m(t)}{dt} \right\|_{-1} + |t_{i'} - t| \left\| \frac{du_{m'}(t)}{dt} \right\|_{-1}. \end{aligned}$$

Combining the above inequality with Lemma 3.1 gives

$$(3.18) \quad \|e_{m,m'}(t) - \bar{e}_{m,m'}(t)\|_{-1} \leq C_1\Delta t_i + C_2\Delta t_{i'}.$$

Using  $-\Delta\rho_{m,m'}(t) = e_{m,m'}(t)$  and (2.3), we have for  $2 \leq \rho < \infty$

$$\begin{aligned} (3.19) \quad &\int |\bar{e}_{m,m'}|^{\rho} dx \leq C\langle \bar{e}_{m,m'}, \Phi(u_i) - \Phi(u_{i'}) \rangle \\ &= \langle e_{m,m'} + \bar{e}_{m,m'} - e_{m,m'}, \Phi(u_i) - \Phi(u_{i'}) \rangle \\ &= \langle \nabla\rho_{m,m'}(t), \nabla(\Phi(u_i) - \Phi(u_{i'})) \rangle \\ &\quad + \langle \bar{e}_{m,m'} - e_{m,m'}, \Phi(u_i) - \Phi(u_{i'}) \rangle \\ &\leq (\|\nabla\rho_{m,m'}(t)\|_2 + \|\bar{e}_{m,m'} - e_{m,m'}\|_{-1}) \\ &\quad \times \|\nabla(\Phi(u_i) - \Phi(u_{i'}))\|_2. \end{aligned}$$

By Lemma 3.1 we have  $\|\nabla(\Phi(u_i) - \Phi(u_{i'}))\|_2 \leq C$ . Therefore, by (3.16), (3.18) and (3.19), we have for small  $\Delta t, \Delta t'$

$$\|\bar{e}_{m,m'}(t)\|_{\rho}^2 \leq C(\Delta t_i + \Delta t_{i'})^{\frac{1}{\rho}}.$$

A similar situation holds for  $1 < \rho < 2$ . By taking  $\frac{\rho}{2}$  roots in (2.4) and using Hölder's inequality and Lemma 3.1 we have

$$\begin{aligned} (3.20) \quad \|\bar{e}_{m,m'}(t)\|_{\rho}^{\rho} &\leq C \left[ \int (\Phi(u_i) - \Phi(u_{i'}))(u_i - u_{i'}) \right]^{\frac{\rho}{2}} (\|u_i\|_{\rho}^{\rho} + \|u_{i'}\|_{\rho}^{\rho})^{\frac{2-\rho}{2}} \\ &\leq C \left[ \int (\Phi(u_i) - \Phi(u_{i'}))(u_i - u_{i'}) \right]^{\frac{\rho}{2}}. \end{aligned}$$

Therefore, by (3.17), (3.18) and Lemma 3.1, we have for small  $\Delta t$  and  $\Delta t'$

$$\begin{aligned} \|\bar{e}_{m,m'}(t)\|_{\rho}^2 &\leq C\langle \Phi(u_i) - \Phi(u_{i'}), u_i - u_{i'} \rangle \\ &\leq \|\nabla(\Phi(u_i) - \Phi(u_{i'}))\|_2 \|\bar{e}_{m,m'}(t)\|_{-1} \\ &\leq C\|\bar{e}_{m,m'} - e_{m,m'} + e_{m,m'}\|_{-1} \\ &\leq C(\Delta t + \Delta t')^{\frac{1}{2}}. \end{aligned}$$

Now we may use the same argument at the end of Theorem 3.1 to prove the result. □

**Lemma 3.3.** *Let  $\{u_i\}_{i=1}^m$  be the sequence generated by (3.1). Then there exists a positive constant  $C = C(\Omega, \rho, f, u_0)$  independent of  $\{t_i\}_{i=0}^m$  such that*

$$\max_{1 \leq i \leq m} \frac{\|u_i - u_{i-1}\|_\rho}{\Delta t^{\frac{1}{\rho}}} \leq C \quad \text{for } 2 \leq \rho < \infty,$$

and

$$\max_{1 \leq i \leq m} \frac{\|u_i - u_{i-1}\|_\rho}{\Delta t^{\frac{1}{2}}} \leq C \quad \text{for } 1 < \rho < 2.$$

*Proof.* Fix  $1 \leq i \leq m$ . Taking  $v = \Phi(u_i) - \Phi(u_{i-1})$  in (3.1) we write

$$\begin{aligned} (3.21) \quad & \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle \\ & + \frac{1}{\rho - 1} \langle \nabla \Phi(u_i), \nabla \Phi(u_i) - \nabla \Phi(u_{i-1}) \rangle \\ & = \langle f_i, \Phi(u_i) - \Phi(u_{i-1}) \rangle. \end{aligned}$$

Subtracting  $\frac{1}{\rho - 1} \langle \nabla \Phi(u_{i-1}), \nabla \Phi(u_i) - \nabla \Phi(u_{i-1}) \rangle$  from both sides of (3.21) and using standard estimates gives

$$\begin{aligned} (3.22) \quad & \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle + \frac{1}{\rho - 1} \|\nabla \Phi(u_i) - \nabla \Phi(u_{i-1})\|_2^2 \\ & \leq \left( C \|f_i\|_2 + \frac{1}{\rho - 1} \|\nabla \Phi(u_{i-1})\|_2 \right) \|\nabla \Phi(u_i) - \nabla \Phi(u_{i-1})\|_2 \\ & \leq \frac{1}{2\varepsilon} \left( C \|f_i\|_2 + \frac{1}{\rho - 1} \|\nabla \Phi(u_{i-1})\|_2 \right)^2 + \frac{\varepsilon}{2} \|\nabla \Phi(u_i) - \nabla \Phi(u_{i-1})\|_2^2. \end{aligned}$$

Choose  $0 < \varepsilon < \frac{2}{\rho - 1}$  and subtract the last term in (3.22) from both sides to get

$$\begin{aligned} (3.23) \quad & \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle \\ & + \left( \frac{1}{\rho - 1} - \frac{\varepsilon}{2} \right) \|\nabla \Phi(u_i) - \nabla \Phi(u_{i-1})\|_2^2 \\ & \leq \frac{1}{2\varepsilon} \left( C \|f_i\|_2 + \frac{1}{\rho - 1} \|\nabla \Phi(u_{i-1})\|_2 \right)^2. \end{aligned}$$

By (3.23) and Lemma 3.1 we have

$$\left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle \leq C.$$

Using the estimates of (2.3) and (2.4) and arguing as in (3.20) for the case  $1 < \rho < 2$  we have

$$\frac{\|u_i - u_{i-1}\|_\rho}{\Delta t_i} \leq C \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle \quad \text{for } 2 \leq \rho < \infty,$$

and

$$\frac{\|u_i - u_{i-1}\|_\rho}{\Delta t_i} \leq C \left\langle \frac{u_i - u_{i-1}}{\Delta t_i}, \Phi(u_i) - \Phi(u_{i-1}) \right\rangle \quad \text{for } 1 < \rho < 2.$$

The conclusion follows by taking appropriate roots and using the fact that  $\Delta t_i \leq \Delta t$ . □

**Theorem 3.2.** *Let  $u$  be the exact solution of (1.1) with initial data satisfying  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . And let  $u_m(t)$  be the semi-discrete solution defined by (3.3), (3.4). Then*

$$\|u(t) - u_m(t)\|_\rho \leq C\Delta t^{\frac{1}{2\rho}} \quad \text{for } 2 \leq \rho < \infty,$$

and

$$\|u(t) - u_m(t)\|_\rho \leq C\Delta t^{\frac{1}{4}} \quad \text{for } 1 < \rho < 2.$$

*Proof.* For  $t \in (t_{i-1}, t_i]$  we have

$$\begin{aligned} \|u(t) - u_m(t)\|_\rho &\leq \|u(t) - \bar{u}_m(t)\|_\rho + \|\bar{u}_m(t) - u_m(t)\|_\rho \\ &= \|u(t) - \bar{u}_m(t)\|_\rho + \left\| \frac{t_i - t}{\Delta t_i} (u_i - u_{i-1}) \right\|_\rho \\ &\leq \|u(t) - \bar{u}_m(t)\|_\rho + \|u_i - u_{i-1}\|_\rho. \end{aligned}$$

The conclusion follows for small  $\Delta t$  by applying Lemma 3.2 and Lemma 3.3 to the above inequality. □

#### 4. ERROR ESTIMATES FOR THE FULLY DISCRETE APPROXIMATION

Let  $\mathcal{T}_h$  be a simplicial subdivision of  $\Omega$  with maximum mesh size  $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$ . In this discretization,  $K$  denotes an  $N$ -simplex,  $\text{diam}(K)$  denotes the diameter of  $K$ , and  $\rho_K$  denotes the radius of the largest closed ball contained in  $\bar{K}$ . We assume that  $\mathcal{T}_h$  is regular, i.e., there exists a constant  $\gamma$  independent of  $h$  such that

$$\max_{K \in \mathcal{T}_h} \frac{\text{diam}(K)}{\rho_K} \leq \gamma.$$

Let  $S_0^h$  be the standard  $C^0$  finite element space in  $H_0^1$  consisting of piecewise polynomials of degree  $r$  in  $\Omega$ . Thus,  $\forall v \in S_0^h$  and  $\forall K \in \mathcal{T}_h, v|_K \in \mathcal{P}_r^N$ , where  $\mathcal{P}_r^N$  denotes the set of  $r$  degree polynomials in  $N$  variables (see Ciarlet [5] for further details).

Let  $\Pi$  be the interpolation operator defined by Scott and Zhang [27, p. 486] which is associated with  $S_0^h$ . This operator differs from standard Lagrange interpolation by using local averaging to generate nodal values for functions in Sobolev spaces which may not be pointwise well defined. We will need an estimate of the interpolation error in Sobolev norms. Toward this end, let  $v \in W^{\ell,p}(\Omega)$ , where  $\ell > \frac{1}{p}$  and  $p > 1$ . Suppose further that  $1 < q < +\infty, 0 \leq m \leq \ell \leq r + 1$ , and there is a constant  $\sigma$  satisfying

$$(4.1) \quad 0 < \sigma \leq \frac{1}{q} - \frac{1}{p} + \frac{\ell - m}{N}.$$

Then we claim

$$(4.2) \quad \|v - \Pi v\|_{W^{m,q}(\Omega)} \leq Ch^{\ell - m + N(\frac{1}{q} - \frac{1}{p})} \|v\|_{W^{\ell,p}(\Omega)}.$$

This estimate is a slightly generalized version of that proved in [27, p. 490] where it is proved for the case  $p = q$ . In Lemma 4.1 we will use (4.2) in the case  $\ell = 2, m = 1$ , and  $q = 2$ . In this setting, our assumption  $\rho^* < \rho$  implies (4.1). We will

only sketch the proof of (4.2). The interested reader is referred to [27] and [7] for further details.

Let  $K \in \mathcal{T}_h$  and write

$$(4.3) \quad \|v - \Pi v\|_{W^{m,q}(K)} \leq C\|v - \tilde{p}\|_{W^{m,q}(K)} + \|\Pi(\tilde{p} - v)\|_{W^{m,q}(K)},$$

for any  $\tilde{p} \in \mathcal{P}_r^N$ . This uses the fact (see [27]) that  $\Pi$  is a projection on  $S_0^h$ . To estimate the first term in (4.3) we will use

$$(4.4) \quad \inf_{\tilde{p} \in \mathcal{P}_r^N} \|v - \tilde{p}\|_{W^{m,q}(K)} \leq Ch^{\ell-m+N(\frac{1}{q}-\frac{1}{p})} |v|_{W^{\ell,p}(S_K)},$$

where  $S_K = \text{int}(\bigcup\{\overline{K_i} \cap \overline{K} \neq \emptyset, K_i \in \mathcal{T}_h\})$ . The estimate in (4.4) follows from Dupont and Scott [7, Theorem 3.2]. Although the explicit power of  $h$  shown in (4.4) is not given in [7], it can be computed in the case when (4.1) holds. The second term in (4.3) is bounded by using [27, Theorem 3.1] and (4.4) to get

$$\|\Pi(\tilde{p} - v)\|_{W^{m,q}(K)} \leq Ch^{\ell-m+N(\frac{1}{q}-\frac{1}{p})} |v|_{W^{\ell,p}(S_K)}.$$

The estimate in (4.2) now follows by taking  $\inf_{\tilde{p} \in \mathcal{P}_r^N}$  on both sides of (4.3) and summing for each  $K \in \mathcal{T}_h$ , as done in [27, p. 490].

For simplicity, from this point on we will assume linear interpolating functions and henceforth  $r = 1$ . Let  $s \geq -1$  and define  $T : H^s \rightarrow H_0^1$  by  $Tg = u$ , where  $u \in H_0^1$  is the unique solution of  $-\Delta u = g$ . Thus,  $T = (-\Delta)^{-1}$ . Similarly, we define  $T_h : H^{-1} \rightarrow S_0^h$  such that  $u = T_h g$  denotes the finite element solution of  $-\Delta u = g$  in  $S_0^h$ . Since  $\|\nabla Tg\|_2^2 = \langle g, u \rangle$ , and  $\|\nabla T_h g\|_2^2 = \langle g, T_h g \rangle$ , we have

$$(4.5) \quad \|\nabla Tg\|_2 \leq \|g\|_{-1} \quad \text{and} \quad \|\nabla T_h g\|_2 \leq \|g\|_{-1}$$

for  $g \in H^s$  for  $s \geq -1$ . We recall the following standard  $L^2$  error estimates [5] in the case  $s = 0$

$$(4.6) \quad \begin{aligned} \|Tg - T_h g\|_2 &\leq Ch^2, \\ \|\nabla Tg - \nabla T_h g\|_2 &\leq Ch. \end{aligned}$$

The constant  $C$  depends on  $\|g\|_2$ . The elliptic projection operator  $P : H_0^1 \rightarrow S_0^h$  is defined by  $P = T_h T^{-1}$ . Thus,  $Tg = u$  implies  $T_h g = Pu$  for any  $g \in H^s$ .

We will also need the following  $L^{\rho'}$  version of the Aubin-Nitsche Lemma for the case  $s = -1$ .

**Lemma 4.1.** *Let  $\rho^* \leq \rho < +\infty$  and suppose  $u \in H_0^1$ . Let  $-\Delta u = g \in H^{-1}$ . Then*

$$\|u - Pu\|_{\rho'} \leq C(\Omega, \|g\|_{-1}, \rho, \gamma) h^{N(\frac{\rho-2}{2\rho})+1}.$$

*Proof.* We begin with  $\|u - Pu\|_{\rho'} = \sup_{f \in L^\rho} \frac{\langle u - Pu, f \rangle}{\|f\|_\rho}$ . Rewrite  $f = -\Delta T f$ , integrate by parts, and use the definition of elliptic projection to obtain

$$\langle u - Pu, f \rangle = \langle \nabla u - \nabla Pu, \nabla T f - \nabla \psi \rangle \quad \text{for all } \psi \in S_0^h.$$

Thus, using (4.5) we have

$$(4.7) \quad \begin{aligned} \|u - Pu\|_{\rho'} &\leq \|\nabla u - \nabla Pu\|_2 \sup_{f \in L^\rho} \frac{\|\nabla T f - \nabla \psi\|_2}{\|f\|_\rho} \\ &\leq C(\Omega, \|g\|_{-1}) \sup_{f \in L^\rho} \frac{\|\nabla T f - \nabla \psi\|_2}{\|f\|_\rho}, \end{aligned}$$

where  $\psi \in S_0^h$  is arbitrary.

From standard regularity theory [12] we have if  $f \in L^\rho$ , then  $Tf \in W^{2,\rho} \cap W_0^{1,\rho}$  and

$$(4.8) \quad \|Tf\|_{W^{2,\rho}} \leq C\|f\|_\rho.$$

Combining (4.8) with the interpolation error estimate (4.2) in the case  $\ell = 2$ ,  $m = 1$ , and  $q = 2$ , we obtain

$$\frac{\|\nabla Tf - \nabla \Pi T f\|_2}{\|f\|_\rho} \leq Ch^{N(\frac{\rho-2}{2\rho})+1}.$$

Writing  $\psi = \Pi T f$  in (4.7) then gives the result. □

Suppose we have a nonincreasing partition  $0 = t_0 < t_1 < t_2 < \dots < t_m = T$  with  $\Delta t_i = t_i - t_{i-1}$ . Recall  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . Define  $U_0 = \Phi^{-1}(P\Phi(u_0))$  and let  $\{U_i\}_{i=1}^m$  be the sequence in  $S_0^h$  defined by

$$(4.9) \quad \left\langle \frac{U_i - U_{i-1}}{\Delta t_i}, V \right\rangle + \frac{1}{\rho - 1} \langle \nabla(|U_i|^{\rho-2}U_i), \nabla V \rangle = \langle f_i, V \rangle, \quad \forall V \in S_0^h.$$

The solutions  $U_i$  exist by the same argument used to show existence of solutions to (3.1). The fully discrete solution is defined by

$$(4.10) \quad U_m(t) = \frac{t - t_{i-1}}{\Delta t_i} U_i + \frac{t_i - t}{\Delta t_i} U_{i-1}, \quad \text{for } t_{i-1} < t \leq t_i, \quad i = 1, \dots, m,$$

$$(4.11) \quad U_m(0) = U_0.$$

**Lemma 4.2.** *Suppose  $1 < \rho < \infty$  and the partition  $\{t_i\}_{i=0}^m$  is a nonincreasing partition. Let  $\{U_i\}_{i=1}^m$  be the sequence generated by (4.9) and  $U_m(t)$  be the corresponding fully discrete solution defined by (4.10) and (4.11). Then there exists a positive constant  $C = C(\Omega, \rho, f, u_0)$  independent of  $\{t_i\}_{i=0}^m$  such that*

$$\begin{aligned} \max_{1 \leq i \leq m} \|U_i\|_\rho &\leq C, \\ \max_{1 \leq i \leq m} \|\nabla \Phi(U_i)\|_2 &\leq C, \end{aligned}$$

and

$$\max_{0 \leq t \leq T} \left\| \frac{dU_m(t)}{dt} \right\|_{-1} \leq C.$$

**Lemma 4.3.** *Let  $\{U_i\}_{i=1}^m$  be the sequence generated by (4.9). Then there exists a positive constant  $C = C(\Omega, \rho, f, u_0)$  independent of  $\{t_i\}_{i=0}^m$  such that*

$$\max_{1 \leq i \leq m} \frac{\|U_i - U_{i-1}\|_\rho}{\Delta t_i^{\frac{1}{\rho}}} \leq C \quad \text{for } 2 \leq \rho < \infty,$$

and

$$\max_{1 \leq i \leq m} \frac{\|U_i - U_{i-1}\|_p}{\Delta t_i^{\frac{1}{2}}} \leq C \quad \text{for } 1 < \rho < 2.$$

The proofs of Lemma 4.2 and Lemma 4.3 are almost identical to those of Lemma 3.1 and Lemma 3.3, respectively, so we omit the details. We now state and prove our main error estimate in  $L^\rho$  norm for the fully discrete case.

**Theorem 4.1.** *Let  $u$  be the exact solution of the Cauchy-Dirichlet problem (1.1). Let  $U_m(t)$ , defined by (4.10) and (4.11), be the fully discrete solution of the problem*

with a nonincreasing partition of  $[0, T]$ . Suppose initial data  $u_0$  satisfies  $\Phi(u_0) \in H_0^1 \cap L^{\rho'}$ . Then

$$\int_0^T \|u(t) - U_m(t)\|_\rho^\rho dt \leq C_1 \Delta t^{\frac{1}{2}} + C_2 h^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} + C_3 h \quad \text{for } 2 \leq \rho < \infty$$

and

$$\int_0^T \|u(t) - U_m(t)\|_\rho^2 dt \leq C_1 \Delta t^{\frac{1}{2}} + C_2 h^{N(\frac{\rho-2}{2\rho})+1} + C_3 h \quad \text{for } \rho^* < \rho < 2,$$

where the constants  $C_1, C_2$ , and  $C_3$  depend only on  $\Omega, T, u_0, \rho, \gamma$ , and  $f$ .

*Proof.* Let  $u(t)$  be the exact solution of (1.1) in the sense of (2.1) and  $u_m(t)$  be the semi-discrete solution defined by (3.3) and (3.4). Then for  $2 \leq \rho < \infty$  we have, by convexity of  $f(x) = |x|^\rho$ ,

$$(4.12) \quad \int_0^T \|u(t) - U_m(t)\|_\rho^\rho dt \leq C \int_0^T \|u(t) - u_m(t)\|_\rho^\rho dt + C \int_0^T \|e_m(t)\|_\rho^\rho dt,$$

where  $e_m(t) = u_m(t) - U_m(t)$ . For  $\rho^* < \rho < 2$  we consider

$$(4.13) \quad \int_0^T \|u(t) - U_m(t)\|_\rho^2 dt \leq C \int_0^T \|u(t) - u_m(t)\|_\rho^2 dt + C \int_0^T \|e_m(t)\|_\rho^2 dt.$$

By Theorem 3.2, the first term in both (4.12) and (4.13) is  $O(\Delta t^{\frac{1}{2}})$ . To estimate the second term, we observe from Hölder's inequality

$$\frac{d}{dt} \|e_m(t)\|_\rho^\rho \leq \rho \|e_m(t)\|_\rho^{\rho-1} \|\dot{e}_m(t)\|_\rho;$$

therefore,

$$(4.14) \quad \frac{d}{dt} \|e_m(t)\|_\rho \leq \|\dot{e}_m(t)\|_\rho.$$

Integrating (4.14) from  $t_{i-1}$  to  $t$  where  $i = 1, \dots, m$ , we obtain

$$\|e_m(t)\|_\rho \leq \|e_m(t_{i-1})\|_\rho + \int_{t_{i-1}}^t \|\dot{e}_m(\tau)\|_\rho d\tau \quad \text{for } t_{i-1} \leq t < t_i.$$

By Lemma 3.3 and Lemma 4.3 we have for  $2 \leq \rho < \infty$  that  $\|\dot{e}_m(t)\|_\rho \leq C(\Delta t)^{\frac{1}{\rho'}}$  and hence

$$\|e_m(t)\|_\rho^\rho \leq C \|e_m(t_{i-1})\|_\rho^\rho + O(\Delta t).$$

Similarly for  $\rho^* < \rho < 2$  we obtain

$$\|e_m(t)\|_\rho^2 \leq C \|e_m(t_{i-1})\|_\rho^2 + O(\Delta t).$$

It remains to estimate  $\int_0^T \|e_m(t_{i-1})\|_\rho^\rho dt$  and  $\int_0^T \|e_m(t_{i-1})\|_\rho^2 dt$  for  $2 \leq \rho \leq \infty$  and  $\rho^* < \rho < 2$ , respectively.

For  $i = 0, \dots, m$ , let  $W_i = \Phi^{-1}(Pv_i)$  where  $v_i = \Phi(u_i)$ , and  $u_i$  is defined by (3.1). Then by the definition of  $P$

$$(4.15) \quad \langle \Delta \Phi(W_i), V \rangle = \langle \Delta \Phi(u_i), V \rangle, \quad \forall V \in S_0^h.$$

For  $2 \leq \rho < \infty$  we write

$$\int_0^T \|e_m(t_{i-1})\|_\rho^\rho dt \leq C \int_0^T \|u_{i-1} - W_{i-1}\|_\rho^\rho dt + C \int_0^T \|W_{i-1} - U_{i-1}\|_\rho^\rho dt$$

and for  $\rho^* < \rho < 2$  we write

$$\int_0^T \|e_m(t_{i-1})\|_\rho^2 dt \leq C \int_0^T \|u_{i-1} - W_{i-1}\|_\rho^2 dt + C \int_0^T \|W_{i-1} - U_{i-1}\|_\rho^2 dt.$$

We begin by estimating the integrand of the first term in each of these expressions. For  $2 \leq \rho < \infty$  we have, by (2.3),

$$\begin{aligned} \|u_{i-1} - W_{i-1}\|_\rho^\rho &\leq C \langle |u_{i-1}|^{\rho-2} u_{i-1} - |W_{i-1}|^{\rho-2} W_{i-1}, u_{i-1} - W_{i-1} \rangle \\ &= C \langle v_{i-1} - P v_{i-1}, u_{i-1} - W_{i-1} \rangle \\ &\leq C \|v_{i-1} - P v_{i-1}\|_{\rho'} \|u_{i-1} - W_{i-1}\|_\rho. \end{aligned}$$

We conclude

$$\|u_{i-1} - W_{i-1}\|_\rho^{\rho-1} \leq C \|P v_{i-1} - v_{i-1}\|_{\rho'} \quad \text{for } i = 1, \dots, m.$$

From Lemma 3.1 we observe that  $v_{i-1} \in H_0^1$ ; therefore,  $-\Delta v_{i-1} \in H^{-1}$ . Apply the estimate from Lemma 4.1 and use  $\rho \geq 2$  to get  $\|P v_{i-1} - v_{i-1}\|_{\rho'} \leq Ch^{N(\frac{\rho-2}{2\rho})+1}$ . Thus

$$(4.16) \quad \|u_{i-1} - W_{i-1}\|_\rho^\rho \leq Ch^{\rho'(N(\frac{\rho-2}{2\rho})+1)}, \quad \text{for } i = 1, \dots, m.$$

For  $\rho^* \leq \rho < 2$  we take  $\frac{\rho}{2}$  roots in (2.4), integrate, and use Hölder’s inequality with  $\beta = \frac{2}{2-\rho}$  and  $\beta' = \frac{2}{\rho}$  to obtain

$$\|u_{i-1} - W_{i-1}\|_\rho^\rho \leq C \langle v_{i-1} - P v_{i-1}, u_{i-1} - W_{i-1} \rangle^{\frac{\rho}{2}} \| |u_{i-1}| + |W_{i-1}| \|_\rho^{\frac{\rho(2-\rho)}{2}}.$$

Thus,

$$(4.17) \quad \begin{aligned} &\|u_{i-1} - W_{i-1}\|_\rho^2 \\ &\leq C \|v_{i-1} - P v_{i-1}\|_{\rho'} \|u_{i-1} - W_{i-1}\|_\rho (\|u_{i-1}\|_\rho + \|W_{i-1}\|_\rho)^{2-\rho}. \end{aligned}$$

We now establish a uniform bound for  $\|W_i\|_\rho$  independent of  $i$ . By Lemma 3.1  $v_{i-1} \in H_0^1$ . Since  $P : H_0^1 \rightarrow S_0^h \subset H_0^1$  we conclude that  $\Phi(W_{i-1}) = P v_{i-1} \in H_0^1$ . By the Sobolev imbedding for  $N \geq 3$  we have

$$(4.18) \quad \|w\|_\rho \leq \|\nabla w\|_2 \quad \text{for } 1 < \rho \leq \frac{2N}{N-2} \quad \text{and } w \in H_0^1.$$

In the case that  $N = 1$  or  $2$ , we note that (4.18) holds for any  $1 < \rho < \infty$ . When  $\rho^* < \rho < \infty$  we have  $1 < \rho' < \frac{2N}{N-2}$ , and we conclude  $\|W_{i-1}\|_\rho = \|\Phi(W_{i-1})\|_{\rho'} \leq C \|\nabla \Phi(W_{i-1})\|_2$ . By taking  $V = \Phi(W_i)$  in (4.15) and using Lemma 3.1 we conclude

$$(4.19) \quad \max_{1 \leq i \leq m} \|W_i\|_\rho \leq C.$$

Thus by (4.17), (4.19), Lemma 3.1 and using (4.18) to estimate  $\|v_{i-1} - P v_{i-1}\|_{\rho'}$ , we obtain

$$\|u_{i-1} - W_{i-1}\|_\rho \leq C \|v_{i-1} - P v_{i-1}\|_{\rho'}.$$

Therefore, by Lemma 4.1

$$(4.20) \quad \|u_{i-1} - W_{i-1}\|_\rho^2 \leq Ch^{2(N(\frac{\rho-2}{2\rho})+1)}.$$

Toward estimating  $\int_0^T \|W_{i-1} - U_{i-1}\|_\rho^\rho dt$  and  $\int_0^T \|W_{i-1} - U_{i-1}\|_\rho^2 dt$ , we observe that by (2.3), (2.4), Lemma 4.2, and (4.19) we have

$$(4.21) \quad \begin{aligned} \|W_i - U_i\|_\rho^\rho &\leq C \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle, \quad \text{for } 2 \leq \rho < \infty, \text{ and} \\ \|W_i - U_i\|_\rho^2 &\leq C \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle, \quad \text{for } \rho^* < \rho < 2. \end{aligned}$$

Thus, it suffices to estimate  $\int_0^T \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle dt$ , which we do as follows.

Let  $\rho_m(t) = Te_m(t)$ . Then, for  $t_{i-1} < t \leq t_i$ , we have

$$\begin{aligned}
 \langle \nabla \dot{\rho}_m(t), \nabla \rho_m(t) \rangle &= \langle -\Delta \dot{\rho}_m(t), \rho_m(t) \rangle \\
 &= \langle \dot{e}_m(t), Te_m(t) \rangle \\
 (4.22) \quad &= \langle \dot{e}_m(t), T(u_m(t) - W_i + U_i - U_m(t)) \rangle \\
 &\quad + \langle \dot{e}_m(t), [T - T_h](W_i - U_i) \rangle \\
 &\quad + \langle \dot{e}_m(t), T_h(W_i - U_i) \rangle.
 \end{aligned}$$

Subtracting (4.9) from (3.1) we have

$$(4.23) \quad \left\langle \frac{du_m(t)}{dt} - \frac{dU_m(t)}{dt}, V \right\rangle + \frac{1}{\rho - 1} \langle \nabla \Phi(u_i) - \nabla \Phi(U_i), \nabla V \rangle = 0, \quad \forall V \in S_0^h.$$

By (4.15), (4.23), integration by parts, and the fact that  $-\Delta : H_0^1 \rightarrow H^{-1}$  is an isometry we get

$$(4.24) \quad \left\langle \frac{du_m(t)}{dt} - \frac{dU_m(t)}{dt}, V \right\rangle + \frac{1}{\rho - 1} \langle \Phi(W_i) - \Phi(U_i), -\Delta V \rangle = 0, \quad \forall V \in S_0^h.$$

In (4.24), let  $V = T_h(W_i - U_i)$ . Then, we have

$$\begin{aligned}
 (4.25) \quad &\langle \dot{e}_m(t), T_h(W_i - U_i) \rangle \\
 &= -\frac{1}{\rho - 1} \langle \Phi(W_i) - \Phi(U_i), (-\Delta)T_h(W_i - U_i) \rangle \\
 &= -\frac{1}{\rho - 1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i + (-\Delta)[T_h - T](W_i - U_i) \rangle.
 \end{aligned}$$

By (4.22) and (4.25) we have

$$(4.26) \quad \frac{1}{\rho - 1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle + \frac{1}{2} \frac{d}{dt} \|\nabla \rho_m(t)\|_2^2 \leq I + II + III,$$

where

$$\begin{aligned}
 I &= \langle \dot{e}_m(t), T(u_m(t) - W_i + U_i - U_m(t)) \rangle, \\
 II &= \langle \dot{e}_m(t), [T - T_h](W_i - U_i) \rangle,
 \end{aligned}$$

and

$$III = \frac{-1}{\rho - 1} \langle \Phi(W_i) - \Phi(U_i), (-\Delta)[T_h - T](W_i - U_i) \rangle.$$

First we obtain a bound for  $I$ . By Lemma 3.1 and Lemma 4.2, we have  $\|\dot{e}_m(t)\|_{-1} \leq C$ . Therefore, using (4.6) we have,

$$\begin{aligned}
 (4.27) \quad |I| &\leq \|\dot{e}_m(t)\|_{-1} \|\nabla T(u_m(t) - W_i + U_i - U_m(t))\|_2 \\
 &\leq \|\dot{e}_m(t)\|_{-1} \|u_m(t) - W_i + U_i - U_m(t)\|_{-1} \\
 &\leq C(\|u_m(t) - W_i\|_{-1} + \|U_i - U_m(t)\|_{-1}) \\
 &\leq C \left( \Delta t \left\| \frac{u_i - u_{i-1}}{\Delta t_i} \right\|_{-1} + \|u_i - W_i\|_{-1} + \Delta t \left\| \frac{U_i - U_{i-1}}{\Delta t_i} \right\|_{-1} \right) \\
 &\leq C_1 \Delta t + C_2 \|u_i - W_i\|_{-1}.
 \end{aligned}$$



To estimate  $\|u_i - W_i\|_{-1}$  we recall that for  $\rho^* < \rho < \infty$  we have  $H_0^1 \subset L^{\rho'}$ ; hence,  $L^\rho \subset H^{-1}$ . By (4.16) and (4.20) we have

$$(4.28) \quad \begin{aligned} \|W_i - u_i\|_{-1} &\leq \|W_i - u_i\|_\rho \leq Ch^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} \quad \text{for } 2 \leq \rho < \infty, \\ \|W_i - u_i\|_{-1} &\leq \|W_i - u_i\|_\rho \leq Ch^{N(\frac{\rho-2}{2\rho})+1} \quad \text{for } \rho^* < \rho < 2. \end{aligned}$$

From (4.27) and (4.28) we conclude

$$(4.29) \quad \begin{aligned} |I| &\leq C_i \Delta t + C_2 h^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} \quad \text{for } 2 \leq \rho < \infty, \text{ and} \\ |I| &\leq C_1 \Delta t + C_2 h^{N(\frac{\rho-2}{2\rho})+1} \quad \text{for } \rho^* < \rho < 2. \end{aligned}$$

An estimate for  $II$  can be established using Lemma 3.1, Lemma 4.1, and the second inequality in (4.5) as follows:

$$(4.30) \quad \begin{aligned} |II| &\leq \|\dot{e}_m(t)\|_{-1} \|\nabla[T - T_h](W_i - U_i)\|_2 \\ &\leq C \|\nabla[T - T_h](W_i - U_i)\|_2 \\ &\leq Ch. \end{aligned}$$

As noted in the remark following (4.5), the constant in (4.30) depends on  $\|W_i - U_i\|_{L^2}$ . We claim  $U_i$  and  $W_i$  are bounded in  $L^2 = H^0$  independent of  $i$ . First consider  $2 \leq \rho$ . By Lemma 4.2,  $U_i \in L^\rho \subset L^2 = H^0$ . Moreover, we note that

$$\|W_i\|_2 \leq C + \|W_i - U_i\|_2 \leq C \|W_i - u_i\|_2 + \|u_i - U_i\|_2,$$

which is bounded by (4.16), Lemma 3.1, and Lemma 4.2. Similarly,  $\rho^* < \rho < 2$  implies  $1 < \frac{2}{\rho-1} < \frac{2N}{N-2}$ . Using the Sobolev imbedding (4.18) we conclude  $\Phi(W_i) \in L^{\frac{2}{\rho-1}}$  and hence  $W_i \in L^2$ . The same argument shows  $U_i \in L^2$ . In both cases we have  $\|W_i - U_i\|_2$  bounded.

We can estimate the third term on the right hand side of (4.26) in a similar manner. Observe using Lemma 3.1, Lemma 4.2, and (4.15) with  $V = \Phi(W_i)$  that

$$(4.31) \quad \begin{aligned} |III| &\leq C \|\nabla(\Phi(W_i) - \Phi(U_i))\|_2 \|\nabla[T_h - T](W_i - U_i)\|_2 \\ &\leq C \|\nabla[T_h - T](W_i - U_i)\|_2 \\ &\leq Ch. \end{aligned}$$

Combining (4.29), (4.30), (4.31), (4.26), and assuming  $0 < h \leq 1$ , we have for  $2 \leq \rho < \infty$

$$(4.32) \quad \begin{aligned} &\frac{1}{\rho-1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle + \frac{1}{2} \frac{d}{dt} \|\nabla \rho_m(t)\|_2^2 \\ &\leq C_1 \Delta t + C_2 h^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} + C_3 h, \end{aligned}$$

and for  $\rho^* < \rho < 2$

$$(4.33) \quad \begin{aligned} &\frac{1}{\rho-1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle + \frac{1}{2} \frac{d}{dt} \|\nabla \rho_m(t)\|_2^2 \\ &\leq C_1 \Delta t + C_2 h^{N(\frac{\rho-2}{2\rho})+1} + C_3 h. \end{aligned}$$

Using (4.5) we have  $\|\nabla \rho_m(0)\|_2 \leq \|e_m(0)\|_{-1} = \|u_0 - U_0\|_{-1} = \|u_0 - W_0\|_{-1} \leq \|u_0 - W_0\|_\rho$ , which is controlled by (4.16) or (4.20). Therefore, integrating (4.32)

and (4.33) over  $[t_{i-1}, t_i]$  and summing on  $i$  gives the following: for  $2 \leq \rho < \infty$ ,

$$(4.34) \quad \sum_{i=0}^m \int_{t_{i-1}}^{t_i} \frac{1}{\rho-1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle dt + \|\nabla \rho_m(t)\|_2 \\ \leq C_1 \Delta t + C_2 h^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} + C_3 h;$$

and for  $\rho^* < \rho < 2$ ,

$$(4.35) \quad \sum_{i=0}^m \int_{t_{i-1}}^{t_i} \frac{1}{\rho-1} \langle \Phi(W_i) - \Phi(U_i), W_i - U_i \rangle dt \\ \leq C_1 \Delta t + C_2 h^{N(\frac{\rho-2}{2\rho})+1} + C_3 h.$$

Combining (4.21) with (4.34) and (4.35) gives

$$\int_0^T \|W_i - U_i\|_\rho^\rho \leq C_1 \Delta t + C_2 h^{\frac{1}{\rho-1}(N(\frac{\rho-2}{2\rho})+1)} + C_3 h \quad \text{for } 2 \leq \rho < \infty$$

and

$$\int_0^T \|W_i - U_i\|_\rho^2 \leq C_1 \Delta t + C_2 h^{N(\frac{\rho-2}{2\rho})+1} + C_3 h \quad \text{for } \rho^* < \rho < 2.$$

This completes our estimates for  $\int_0^T \|e_m(t_{i-1})\|_\rho^\rho dt$  and  $\int_0^T \|e_m(t_{i-1})\|_\rho^2 dt$ . Thus, we have established Theorem 4.1.  $\square$

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#### REFERENCES

1. J. G. Berryman and C. J. Holland, *Nonlinear diffusion problem arising in plasma physics*, Phys. Rev. Lett. **40** (1978), 1720–1722. MR **58**:14366
2. H. Brézis and M. Crandall, *Uniqueness of solutions of the initial value problem for  $u_t - \Delta \phi(u) = 0$* , J. Math. Pures Appl. **58** (1979), 153–163. MR **80e**:35029
3. H. Brézis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. **62** (1983), 73–97. MR **84g**:35093
4. F. E. Browder, *Existence and uniqueness theorems for solutions of nonlinear boundary value problems*, in Proceedings of Symposia in Applied Mathematics 17, Amer. Math. Soc., Providence, RI, 1964. MR **33**:6092
5. P. G. Ciarlet and J. L. Lions (eds.), *Handbook of Numerical Analysis, Volume II: Finite Element Methods (Part 1)*, North-Holland, 1991. MR **92f**:65001
6. E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, Berlin, 1994. MR **94h**:35130
7. T. DuPont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp. **34** (1980), 441–463. MR **81h**:65014
8. A. Eden, B. Michaux, and J. M. Rakotoson, *Semi-discretized nonlinear evolution equations as discrete dynamical systems and error analysis*, Indiana U. Math. J. **39** (1990), 737–783. MR **91h**:35150
9. C. M. Elliott, *Error analysis of the enthalpy method for the Stefan problem*, IMA J. of Num. Anal. **7** (1987), 61–71. MR **90a**:65222
10. I. Farago, *Finite element method for solving nonlinear parabolic systems*, Computers & Math. with Appl. **21** (1991), 49–59.

11. S. M. F. Garcia, *Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: The discrete-time case*, Num. Meth. for Part. Diff. Eq. **10** (1994), 149–170. MR **95a**:65164
12. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, New York, 1983. MR **86c**:35035
13. J. R. King, *Self-similar behaviour for the equation of fast nonlinear diffusion*, Phil. Trans. Royal Soc. Lond. A **343** (1993), 337–375.
14. J. W. Jerome and M. E. Rose, *Error estimates for the multidimensional two-phase Stefan problem*, Math. Comp. **39** (1982), 377–414. MR **84h**:65097
15. A. S. Kalashnikov, *Some problems of the qualitative theory of the non-linear degenerate second order parabolic equations*, Russian Math. Surveys **42** (1987), 169–222.
16. M.-N. Le Roux, *Semidiscretization in time of a fast diffusion equation*, J. Math. Anal. Appl. **137** (1989), 354–370. MR **90k**:65166
17. P. Lesaint and J. Pousin, *Error estimates for a nonlinear degenerate parabolic problem*, Math. Comp. **59** (1992), 339–358. MR **93a**:35094
18. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969. MR **41**:4326
19. R. H. Nochetto, *Error estimates for multidimensional singular parabolic problems*, Japan J. Appl. Math. **4** (1987), 111–138. MR **89c**:65107
20. R. H. Nochetto and C. Verdi, *Approximation of degenerate parabolic problems using numerical integration*, SIAM J. Numer. Anal. **25** (1988), 784–814. MR **89m**:65102
21. R. H. Nochetto, personal communication.
22. P. A. Raviart, *Sur la résolution de certaines équations paraboliques non linéaires dégénérées*, J. Func. Anal. **5** (1970), 299–328. MR **41**:2235
23. P. A. Raviart, *Sur la résolution et l'approximation de certaines équations paraboliques non linéaires dégénérées*, Arch. Rat. Mech. Anal. **25** (1967), 64–80. MR **35**:6384
24. M. E. Rose, *Numerical methods for flows through porous media. I*, Math. Comp. **40** (1983), 435–467. MR **85a**:65146
25. J. Rulla, *Error analysis for implicit approximations to solutions to Cauchy problems*, SIAM J. on Numer. Anal. **33** (1996), 68–87. MR **97c**:65151
26. J. Rulla and N. Walkington, *Optimal rates of convergence for degenerate parabolic problems in two dimensions*, SIAM J. on Numer. Anal. **33** (1996), 56–67. MR **97c**:65150
27. L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493. MR **90j**:65021
28. M. Tsutsumi, *On solutions of some doubly nonlinear degenerate parabolic equations with absorption*, J. Math. Anal. and Appl. **132** (1988), 187–212. MR **89k**:35118
29. C. Verdi, *Optimal error estimates for an approximation of degenerate parabolic problems*, Numer. Funct. Anal. Optim. **9** (1987), 657–670. MR **88m**:65165
30. A. Ženíšek, *Nonlinear elliptic and evolution problems and their finite element approximations*, Academic Press, New York, 1990. MR **92c**:65003

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